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ON A CONVOLUTION THEOREM FOR $L(p, q)$ SPACES

BY
A. P. BLOZINSKI

Abstract. The principal result of this paper is a proof of the Convolution Theorem based on the definition of a convolution operator as presented by E. M. Stein and R. O'Neil. Closely related are earlier versions and special cases of the Convolution Theorem, which are $L(p, q)$ analogues of an inequality of W. H. Young, given in papers by R. O'Neil, L. Y. H. Yap, R. Hunt, and B. Muckenhoupt and E. M. Stein.

1. Introduction.

1.1. The main purpose of this paper is to give a proof of a generalized convolution theorem for $L(p, q)$ spaces. This result was first presented by R. O'Neil [3, Theorem 2.6]. L. Y. H. Yap [4] observed that the proof of the Convolution Theorem, as first presented, was incomplete. Some of the inequalities arrived at in the proof of a basic lemma [3, Lemma 1.5] did not follow without first assuming a strong continuity condition of the convolution operator. For this reason, Yap [4] proposed an alternative definition of a convolution operator (called *positive convolution operator* [4, Definition 2.3.]) and then showed that it could be extended in such a manner that the desired continuity property was available.

We consider those spaces $L(p, q)$ for which $f \in L(p, q)$ implies f is locally integrable (i.e. $f \in L(p, q)$, $1 < p < \infty, 0 < q \leqq \infty; p = \infty, q = \infty; p = q = 1$). Our idea is to work with simple functions. The simple functions form a dense subset of $L(p, q)$, $q \neq \infty$, with respect to the metric of $L(p, q)$ (see Hunt [1]). We restrict O'Neil's definition of a convolution operator T (see Definition 2.1) to simple functions and show that there is a unique bilinear extension of T so that the convolution theorem (see Theorems 2.9, 2.10, 2.12) is valid for those cases where an $L(p, \infty)$, $1 < p < \infty$, space is not involved. This being done, it is shown that those cases where an $L(p, \infty)$, $1 < p < \infty$, space is involved follow. This is possible since $f \in L(p, \infty)$, $1 < p < \infty$, implies f can be written as $f = f_1 + f_2$, where $f_1 \in L(p_1, q_1)$, $f_2 \in L(p_2, q_2)$, $1 < p_1 < p < p_2 < \infty$, $0 < q_1 < \infty$, $0 < q_2 < \infty$. In doing this we make no requirement that such a representation of the function f be unique.

The proof is constructive and is carried out in several steps. Some of the steps are stated as lemmas. For the crucial Lemma 2.2 we borrow from a very elementary and straightforward proof of L. Y. H. Yap [4, Lemma 2.7].

We remark that in [2, p. 77] B. Muckenhoupt and E. M. Stein somewhat anticipated this approach. By a different proof but using the basic ideas of working with

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simple functions and of splitting $L(p, \infty)$ functions, they presented a special case of the Convolution Theorem. Using our notation this theorem states:

If T is a convolution operator and $f \in L(p, p)$, $g \in L(q, \infty)$ where $1 < p, q, r < \infty$ and $1/r = 1/q + 1/p - 1$, then we have after extending T that $T(f, g) \in L(r, r)$. Moreover $\|T(f, g)\|_{(r,r)} \leqq A(p, q)\|f\|_{(p,p)}\|g\|_{(q,\infty)}$, where $A(p, q)$ depends only on p, q.

It is of interest to mention that the Convolution Theorem as we state it (Theorems 2.9, 2.10, 2.12) is the most general possible for $L(p, q)$ spaces. That is, if the given indices p_1, q_1, p_2, q_2 are not related as in these theorems (2.9, 2.10, 2.12), then the convolution operator T need not be continuous on $L(p_1, q_1) \times L(p_2, q_2)$. In fact there may exist $f \in L(p_1, q_1)$, $g \in L(p_2, q_2)$ for which $T(f, g)$ is not defined (this result will appear in a subsequent paper).

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1.2. *Rearrangement functions and $L(p, q)$ spaces.* For the convenience of the reader and the sake of comparison with [1], [3], [4], we give a number of definitions and record some known facts about rearrangements of functions and $L(p, q)$ spaces which will be needed in the sequel. For notation and terms not explained in this paper we refer the reader to R. A. Hunt [1], where a study of the $L(p, q)$ spaces is given in some detail.

We consider complex valued measurable functions f defined on a measure space (X, μ). The measure μ is assumed to be nonnegative. We assume the functions f are finite valued a.e. and for some $y > 0$, $\mu(E_y) < \infty$, where $E_y = E_y[f] = \{x \in X : |f(x)| > y\}$. When it is desirable to suppress mention of the underlying measure we will sometimes set $\mu(E) = |E|$, where $E \subset X$ is a measurable set.

The *distribution function* of f is defined by

$$\lambda_f(y) = \mu(E_y), \qquad y > 0.$$

The *nonnegative rearrangement* of f on $(0, \infty)$ is defined by

$$f^{**}(t) = \inf\{y > 0 : \lambda_f(y) \leqq t\}, \qquad t > 0.$$

The *average function* of f on $(0, \infty)$ is given by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds, \qquad t > 0.$$

Note that $\lambda_f(\cdot), f^*(\cdot), f^{**}(\cdot)$ are nonincreasing and right continuous. Moreover, if f, g are measurable functions, then $(f+g)^{**} \leqq f^{**} + g^{**}$.

By a *simple function* we mean a function f which can be written in the form $f(x) = \sum_{j=1}^N C_j \chi_{E_j}(x)$, where $C_1, \ldots, C_N$ are complex numbers, $E_1, \ldots, E_N$ are sets of finite measure and $\chi_E(x)$ denotes the characteristic function of the measurable set E.

The next two lemmas will be useful later. The first is a consequence of the definitions and is easy to show. We omit the easy proof. A proof of the second lemma is given in [1, p. 252].

LEMMA 1.1. *If f is a measurable function for which $\lambda_f(y)$ is finite on* $(0, \infty)$, *then there exists a sequence of simple functions $(f_k)_{k=1}^{\infty}$ such that*

(i) *$f_k \to f$ pointwise everywhere,*

(ii) $|f_k| \leqq |f_{k+1}| \leqq |f|$, $k=1, 2, \ldots$,

(iii) *$f_k^* \uparrow f^*$ pointwise everywhere,*

(iv) *$f_k^{**} \uparrow f^{**}$ pointwise everywhere.*

LEMMA 1.2. *If $f(x)$ is a nonnegative simple function, then we can write $f(x) = \sum_{j=1}^{N} f_j(x)$, where each $f_j(x)$ is a nonnegative function with exactly one positive value and $f^*(t) = \sum_{j=1}^{N} f_j^*(t)$, $t>0$.*

The *Lorentz space* $L(p, q)$ is the collection of all f such that $\|f\|^*_{(p,q)} < \infty$, where

$$\|f\|^*_{(p,q)} = \left(\int_0^\infty [t^{1/p} f^*(t)]^q \frac{dt}{t}\right)^{1/q}, \qquad 1 < p < \infty, 0 < q < \infty; p = q = 1.$$

$$\|f\|^*_{(p,\infty)} = \sup_{t>0} t^{1/p} f^*(t), \qquad 1 < p \leqq \infty.$$

Let $\|f\|_{(p,q)} = \|f^{**}\|^*_{(p,q)}$, $1<p<\infty, 0<q\leqq\infty$. If $1\leqq p\leqq\infty$, then $\|f\|^*_{(p,p)} = \|f\|_p$, where $\|\cdot\|_p$ is the classical L^p norm. For convenience we shall often use the notation $\|\cdot\|_1$, L^1 instead of $\|\cdot\|^*_{(1,1)}$, $L(1, 1)$ and $\|\cdot\|_\infty$, L^∞ instead of $\|\cdot\|^*_{(\infty,\infty)}$, $L(\infty, \infty)$.

LEMMA 1.3. $\|f\|^*_{(p,q)} \leqq \|f\|_{(p,q)} \leqq C(p, q)\|f\|^*_{(p,q)}$, $1<p<\infty$, $0<q\leqq\infty$, *where $C(p, q)$ depends only on p, q.*

Lemma 1.3 above allows us to compare the theorems given in this paper to the theorems given in Hunt [1], O'Neil [3] and Yap [4]. A proof for the case $1<p<\infty$, $1\leqq q\leqq\infty$ is given in [1] and a proof for the case $1<p<\infty$, $0<q<1$ is given in [4].

From the definition of $\|\cdot\|^*_{(p,q)}$, it follows that if $f \in L(p, q)$, $1<p<\infty$, $0<q\leqq\infty$, $p=q=1$, then the function $\lambda_f(y)$ is finite valued on $(0, \infty)$. In this case, with a little thought, it is easy to see that it is possible to construct a sequence of functions $(f_k)_{k=1}^{\infty}$ which satisfy Lemma 1.1 and have the additional property that $(f-f_k)^*(t)$ and $(f-f_k)^{**}(t)$ converge to zero pointwise everywhere, as k tends to infinity. The existence of such a sequence will be useful later.

The remarks above and the Lebesgue dominated convergence theorem imply the following lemma.

LEMMA 1.4. *Simple functions are dense in $L(p, q)$, $q\neq\infty$.*

The next lemma tells us how $L(p, q)$ spaces which have the same first index are related to each other (see [1, p. 253] for a proof).

LEMMA 1.5.

$$\|f\|^*_{(p,s)} \leqq (p/s)^{1/s}(q/p)^{1/q}\|f\|^*_{(p,q)}, \qquad 1 < p < \infty, 0 < q \leqq s < \infty,$$

$$\|f\|^*_{(p,\infty)} \leqq (q/p)^{1/q}\|f\|^*_{(p,q)}, \qquad 1 < p < \infty, 0 < q < \infty.$$

This clearly implies $L(p,q) \subset L(p,s)$, $1<p<\infty$, $0<q\leqq s\leqq\infty$. Since $f^{**}=(f^{**})^*$, $\|\cdot\|^*_{(p,q_i)}$ can be replaced by $\|\cdot\|_{(p,q_i)}$, $q_i=s, q, \infty$, in the statement of Lemma 1.3. We remark that the constants involved above are of the order $C(q,s)=O(q^{1/q-1/s})$, where $C(\infty,\infty)=O(1)$.

$L(p,q)$ spaces can be defined on a wider range of indices p, q (see [1]). Our purpose in this paper is to investigate an abstract notion of convolution of functions defined on two arbitrary measure spaces. In particular, this generalization, and the theorems which follow, will include ordinary convolution of functions defined on Euclidean n-space E^n. In the latter case functions which do not belong to one of the spaces $L(p,q)$, to which we have restricted ourselves, are not appropriate for ordinary convolution since they are not necessarily locally integrable (an exception is the space $L(\infty,q)$, $0<q<\infty$, which is not of interest since $\|f\|^*_{(\infty,q)}<\infty$ implies $f=0$ a.e.).

2. **Convolution operators.**

2.1. *Definition and basic lemmas.*

DEFINITION 2.1. Given three measure spaces (X,μ), $(\bar{X},\bar{\mu})$ and (Y,ν), a bilinear operator T which maps all pairs of simple functions on X and $\bar{X}$ into measurable functions on Y is a convolution operator if for simple functions f, f_1, f_2, g, g_1, g_2 we have

(i) $\|T(f,g)\|_1 \leqq \|f\|_1\|g\|_1$,

(ii) $\|T(f,g)\|_\infty \leqq \|f\|_\infty\|g\|_1$,

(iii) $\|T(f,g)\|_\infty \leqq \|f\|_1\|g\|_\infty$,

(iv) $T(f_1+f_2,g)=T(f_1,g)+T(f_2,g)$, $T(f,g_1+g_2)=T(f,g_1)+T(f,g_2)$.

For the rest of this paper we shall no longer explicitly mention the underlying measure spaces (X,μ), $(\bar{X},\bar{\mu})$ and (Y,ν) of the convolution operator T. And to simplify the discussion we will assume that the functions defined on the underlying measure spaces X and $\bar{X}$ are real valued.

LEMMA 2.2. *Let T be a convolution operator and suppose $h=T(f,g)$, where f, g are simple functions, then for $t>0$,*

$$h^{**}(t) \leqq 4\left(tf^{**}(t)g^{**}(t)+\int_t^\infty f^*(u)g^*(u)\,du\right).$$

Proof. (I) If $f=\alpha\chi_E$, $g=\beta\chi_F$, where $\alpha,\beta\geqq 0$, then by the linearity of T we may suppose $\alpha=\beta=1$. It is easy to see that

$$f^*(u) = 1 \quad \text{if } u \leqq |E|, \qquad f^{**}(u) = 1 \quad \text{if } u \leqq |E|,$$

$$= 0 \quad \text{if } u > |E|, \qquad = \frac{|E|}{u} \quad \text{if } u > |E|,$$

and similarly for $g^*(u)$ and $g^{**}(u)$. We can assume without loss of generality that $|E| \leqq |F|$ and consider two subcases:

(a) If $t > |F|$, then

$$tf^{**}(t)g^{**}(t) + \int_t^\infty f^*(u)\, du = \frac{|E|\,|F|}{t}.$$

But $th^{**}(t) = \int_0^t h^*(u)\, du \leqq \|h\|_1 \leqq \|f\|_1 \|g\|_1 = |E|\,|F|$. Therefore

$$h^{**}(t) \leqq tf^{**}(t)g^{**}(t) + \int_t^\infty f^*(u)g^{**}(u)\, du.$$

(b) If $|E| \leqq t \leqq |F|$ or $t \leqq |E|$, then

$$tf^{**}(t)g^{**}(t) + \int_t^\infty f^*(u)g^*(u)\, du = |E|.$$

But $h^{**}(t) \leqq \|h\|_\infty \leqq \|f\|_1 \|g\|_\infty = |E|$. Therefore

$$h^{**}(t) \leqq tf^{**}(t)g^{**}(t) + \int_t^\infty f^*(u)g^*(u)\, du.$$

(II) If f, g are positive simple functions, then we can write f and g as in Lemma 1.2, so that $T(f, g) = \sum_{i,j} T(f_i, g_j)$. By the triangle inequality and (I) we have

$$\begin{aligned} h^{**}(t) &\leqq \sum_{i,j} [T(f_i, g_j)]^{**}(t) \leqq \sum_{i,j} \left(tf_i^{**}(t)g_j^{**}(t) + \int_t^\infty f_i^*(u)g_j^*(u)\, du \right) \\ &= tf^{**}(t)g^{**}(t) + \int_t^\infty f^*(u)g^*(u)\, du. \end{aligned}$$

If f, g are arbitrary simple functions we can write f and g in terms of their positive and negative components. Each component is clearly a nonnegative simple function. We have $T(f, g) = T(f^+, g^-) - T(f^-, g^-) - T(f^+, g^-) + T(f^-, g^-)$.

If f is a measurable function, then $(f^\pm)^* \leqq f^*$. Using this together with the triangle inequality, the rest of the proof is immediate.

In the sequel the symbol C will be used generically for absolute constants such as the one appearing in Lemma 2.2. The next lemma will be useful later (see O'Neil [3, p. 133] for a proof).

Lemma 2.3. *If f, g are measurable functions, then*

$$tf^{**}(t)g^{**}(t) + \int_t^\infty f^*(u)g^*(u)\, du \leqq \int_t^\infty f^{**}(u)g^{**}(u)\, du.$$

2.4. *Convolution theorems.*

Theorem 2.4. *Let T be a convolution operator and $h = T(f, g)$. T can be uniquely extended so that if $f \in L(p, q)$, $1 < p < \infty$, $q \neq \infty$ and $g \in L^1$, then $h \in L(p, s)$, where $q \leqq s$.*

Moreover

(1) $h^{**}(t) \leqq C(tf^{**}(t)g^{**}(t)+\int_t^\infty f^*(u)g^*(u)\,du)$.

(2) $\|h\|_{(p,s)} \leqq C(q,s)\|f\|_{(p,q)}\|g\|_1$, *where* $C(q,s)=O(q^{1/q-1/s})$.

Proof. Since $q \leqq s$ implies $\|h\|_{(p,s)} \leqq C(q,s)\|h\|_{(p,q)}$, it is sufficient to assume $s=q$.

(I) Let f, g be simple functions. (1) follows from Lemma 2.2. We have

$$\begin{aligned} h^{**}(t) &\leqq C\left(tf^{**}(t)g^{**}(t)+\int_t^\infty f^*(u)g^*(u)\,du\right) \\ &\leqq C\left(tf^{**}(t)g^{**}(t)+f^{**}(t)\int_t^\infty g^*(u)\,du\right) \\ &= Cf^{**}(t)\|g\|_1. \end{aligned}$$

Therefore $\|h\|_{(p,q)} \leqq C\|f\|_{(p,q)}\|g\|_1$.

(II) Let $f \in L(p,q)$ and g be a simple function. For g fixed and all simple functions F define the continuous linear operator $T_g: L(p,q)|_S \to L(p,q)$ by $T_g(F)=T(F,g)$, where S denotes the simple functions. The simple functions are dense in $L(p,q)$. Therefore T_g may be extended to a unique continuous linear operator on all of $L(p,q)$ so that (2) holds.

For the proof of (1) we proceed as follows. Let $(f_k)_{k=1}^\infty$ be a sequence of simple functions, as in Lemma 1.1, which converge to f in the metric of $L(p,q)$. We have

$$\begin{aligned} &\left(\int_0^\infty [t^{1/p-1/q}|T(f_k,g)^{**}(t)-T(f,g)^{**}(t)|]^q\,dt\right)^{1/q} \\ &\qquad \leqq \left(\int_0^\infty [t^{1/s}T(f_k-f,g)^{**}(t)]^q\,\frac{dt}{t}\right)^{1/q} \\ &\qquad = \|T(f_n-f,g)\|_{(p,q)} \leqq C\|f_n-f\|_{(p,q)}\|g\|_1. \end{aligned}$$

The last term converges to zero as k tends to infinity. Passing to a subsequence (k_i) of (k) and using the monotone convergence theorem we have for a.e. t

$$\begin{aligned} T(f,g)^{**}(t) &= \lim_{i\to\infty} T(f_{k_i},g)^{**}(t) \\ &\leqq \lim_{i\to\infty} C\left(tf_{k_i}^{**}(t)g^{**}(t)+\int_t^\infty f_{k_i}^*(u)g^*(u)\,du\right) \\ &= C\left(tf^{**}(t)g^{**}(t)+\int_t^\infty f^*(u)g^*(u)\,du\right). \end{aligned}$$

This establishes (1).

(III) If $f \in L(p,q)$ and $g \in L^1$ repeat the argument of (II), this time fixing $f \in L(p,q)$. The convolution operator T is clearly bilinear on $L(p,q)\times L^1$. To allow $q=\infty$ in Theorem 2.4 we shall need a definition and several lemmas.

LEMMA 2.5. *Let* $f \in L(p,\infty)$, $1<p<\infty$. *Put*

$$\begin{aligned} f^v(x) &= f(x) \quad \text{if } |f(x)| > f^*(v), \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

and $f_v(x)=f(x)-f^v(x)$, *where* $0<v<\infty$. *If* $1<p_1<p<p_2<\infty$, *then*

(1) $f^v \in L(p_1, q_1)$, $0<q_1 \leqq \infty$,

(2) $f_v \in L(p_2, q_2)$, $0<q_2 \leqq \infty$.

Proof. The proof follows by a straightforward calculation using the definition of $\|\cdot\|^*_{(p,q)}$ and Lemma 1.5. We omit the easy details.

This allows us to make the following definition.

DEFINITION 2.6. Let $f \in L(p, \infty)$, $1<p<\infty$, and g be a simple function. If T is a convolution operator we define $T(f, g)=T(f_1, g)+T(f_2, g)$, where $f=f_1+f_2$, $f_1 \in L(p_1, q_1)$, $f_2 \in L(p_2, q_2)$, $1<p_1<p<p_2<\infty$, $0<q_1<\infty$, $0<q_2<\infty$.

By Lemma 2.5 and Theorem 2.4 choices of f_1, f_2 exist so that $T(f, g)$ is defined. But before proceeding we need to show that $T(f, g)$ is the same function independent of any particular choices of f_1, f_2 which satisfy the definition. This follows immediately from the next lemma.

LEMMA 2.7. *Let* $f \in L(p, \infty)$, $1<p<\infty$ *and* g *be a simple function. Suppose* $f=f_1+f_2$, *where* $f_1 \in L(p_1, q_1)$, $f_2 \in L(p_2, q_2)$, $1<p_1<p<p_2<\infty$, $0<q_1<\infty$, $0<q_2<\infty$. *If* T *is a convolution operator and* $0<v<\infty$, *then*

$$T(f_1, g)+T(f_2, g) = T(f^v, g)+T(f_v, g).$$

Proof. Let $E=\{x \in X : |f(x)|>v\}$ and let χ^v, χ_v denote the characteristic functions of the sets E, $X-E$ respectively. We have

$$\begin{aligned} T(f^v, g)+T(f_v, g) &= T(\chi^v f, g)+T(\chi_v f, g) \\ &= T(\chi^v f_1+\chi^v f_2, g)+T(\chi_v f_1+\chi_v f_2, g). \end{aligned}$$

$f^v, \chi^v f_1 \in L(p_1, q_1)$ implies $\chi^v f_2 \in L(p_1, q_1)$ and $f_v, \chi_v f_2 \in L(p_2, q_2)$ implies $\chi_v f_1 \in L(p_2, q_2)$. It follows by the linearity of T that

$$\begin{aligned} T(f^v, g)+T(f_v, g) &= T(\chi^v f_1, g)+T(\chi^v f_2, g)+T(\chi_v f_1, g)+T(\chi_v f_2, g) \\ &= T(\chi^v f_1+\chi_v f_1, g)+T(\chi^v f_2+\chi_v f_2, g) = T(f_1, g)+T(f_2, g). \end{aligned}$$

The next step is to show: If T is a convolution operator, then T is bilinear on $L(p, \infty)\times S$, $1<p<\infty$, $S=\{\text{simple functions}\}$. Using Theorem 2.4 and Definition 2.6, the proof is straightforward. We omit the details.

LEMMA 2.8. *Let* $f \in L(p, \infty)$, $1<p<\infty$, *and* g *be a simple function. If* T *is a convolution operator, then*

$$T(f, g)^{**}(t) \leqq C\left(tf^{**}(t)g^{**}(t)+\int_t^\infty f^*(u)g^*(u)\,du\right), \qquad t>0.$$

Proof. Let $0<v<\infty$. Using Definition 2.6, Theorem 2.4 and the triangle inequality we have

$$\begin{aligned} T(f,g)^{**}(t) &\leqq T(f^v,g)^{**}(t)+T(f_v,g)^{**}(t) \\ &\leqq C\Big\{t(f^v)^{**}(t)g^{**}(t)+\int_t^\infty (f^v)^*(u)g^*(u)\,du \\ &\qquad + t(f_v)^{**}(t)g^{**}(t)+\int_t^\infty (f_v)^*(u)g^*(u)\,du\Big\} \\ &\leqq C\Big\{tf^{**}(t)g^{**}(t)+\int_t^\infty f^*(u)g^*(u)\,du \\ &\qquad + tf^{**}(v)g^{**}(t)+\int_t^\infty f^*(v)g^*(u)\,du\Big\}. \end{aligned}$$

The last two terms are majorized by $f^{**}(v)\|g\|_1$, which converges to zero as v tends to infinity. This implies

$$T(f,g)^{**}(t) \leqq C\Big(tf^{**}(t)g^{**}(t)+\int_t^\infty f^*(u)g^*(u)\,du\Big).$$

We can now state Theorem 2.4 in its general form.

THEOREM 2.9. *Let T be a convolution operator and $h=T(f,g)$. T can be uniquely extended so that*

(1) *If $f\in L(p,q)$, $1<p<\infty$, $g\in L^1$, then $h\in L(p,s)$, where $s\geqq q$. Moreover $\|h\|_{(p,s)}\leqq C(q,s)\|f\|_{(p,q)}\|g\|_1$, where $C(q,s)=O(q^{1/q-1/s})$.*

(2) *If $f\in L^1$, $g\in L(p,q)$, $1<p<\infty$, then $h\in L(p,s)$, where $s\geqq q$. Moreover $\|h\|_{(p,s)}\leqq C(q,s)\|f\|_1\|g\|_{(p,q)}$, where $C(q,s)=O(q^{1/q-1/s})$.*

In either case

(3) $h^{**}(t)\leqq C(tf^{**}(t)g^{**}(t)+\int_t^\infty f^*(u)g^*(u)\,du)$.

Proof. In view of Theorem 2.4 and the symmetry of the arguments it is enough to consider (1) when $q=\infty$ and (3) for this special case.

(I) Let $f\in L(p,\infty)$, $1<p<\infty$, and g be a simple function. We have already shown that the convolution operator T can be extended to a bilinear operator on $L(p,\infty)\times S$, $S=$\{simple functions\}. By Lemma 2.8 we have (3), and so

$$h^{**}(t) \leqq C\Big(tf^{**}(t)g^{**}(t)+\int_t^\infty f^*(u)g^*(u)\,du\Big) \leqq Cf^{**}(t)\|g\|_1.$$

Therefore

$$\|h\|_{(p,\infty)} \leqq C(\infty,\infty)\|f\|_{(p,\infty)}\|g\|_1,$$

which proves (1).

(II) Let $f\in L(p,\infty)$ and $g\in L^1$. The simple functions are dense in L^1. To show (1) and (3) proceed as in part (II) of the proof of Theorem 2.4. This time fix $f\in L(p,\infty)$ and define a linear operator T_f on all simple functions G by $T_f(G)=T(f,G)$. As before T_f extends to a unique linear operator defined on all of L^1 so that (1) holds. (3) follows just as easily. We omit the details.

We complete the proof by remarking that the extended convolution operator T is clearly bilinear on $L(p,\infty)\times L^1$ into $L(p,\infty)$.

THEOREM 2.10. *Let T be a convolution operator and $h=T(f,g)$. T can be uniquely extended so that if $f\in L(p,q_1)$, $1<p<\infty$, and $g\in L(p',q_2)$, where $1/p+1/p'=1$, $1/q_1+1/q_2\geqq 1$, then $h\in L^\infty$. Moreover*

(1) $h^{**}(t)<C(tf^{**}(t)g^{**}(t)+\int_t^\infty f^*(u)g^*(u)\,du)$,

(2) $\|h\|_\infty\leqq C(q_1,q_2)\|f\|_{(p,q_1)}\|g\|_{(p',q_2)}$,

where $C(q_1,q_2)=O(q_1^{1/q_1-1/mq_1})O(q_2^{1/q_2-1/mq_2})$, $m=1/q_1+1/q_2$.

Proof. If $1/q_1+1/q_2=m\geqq 1$, then $1/mq_1+1/mq_2=1$. Since

$$\|f\|_{(p,mq_1)}\|g\|_{(p',mq_2)}\leqq C(q_1,q_2)\|f\|_{(p,q_1)}\|g\|_{(p',q_2)},$$

it is sufficient to assume $1/q_1+1/q_2=1$.

The rest of the proof proceeds in the same way as in Theorems 2.4, 2.9, only this time after establishing (1) we use Lemma 2.3 and Hölder's inequality in order to prove (2).

Before stating the next theorem we need the following lemma.

LEMMA 2.11. *If $1<r<\infty$, $0<q<\infty$ and $f\geqq 0$ is a nonincreasing real valued function on $(0,\infty)$, then*

$$\left(\int_0^\infty [t^{1/r}\int_t^\infty f(u)\,du]^q\frac{dt}{t}\right)^{1/q}\leqq C(r,q)\left(\int_0^\infty [t^{(1/r)+1}f(t)]^q\frac{dt}{t}\right)^{1/q},$$

where $C(r,q)=r$, if $1\leqq q<\infty$ and $C(r,q)\leqq O(2^{q/r}-1)^{-1/q}$, if $0<q<1$.

Proof. The case $1\leqq q<\infty$ is just a special version of Hardy's inequality (see Hunt [1, p. 256] for a proof).

Suppose $0<q<1$. Choose $0<\beta<1$. With two change of variables we have

$$\begin{aligned}\int_0^\infty\left[t^{1/r}\int_t^\infty f(u)\,du\right]^q\frac{dt}{t}&=\int_0^\infty t^{-(q/r)-1}\left[\int_0^t f(1/u)\frac{du}{u^2}\right]^q dt\\&=\int_0^\infty t^{(-q/r)-1}\left[\sum_{n=0}^\infty\int_{\beta^{n+1}}^{\beta^n t} f(1/u)\frac{du}{u^2}\right]^q dt\end{aligned}$$

using the subadditivity of the function $\varphi(x)=x^q$ and the monotonicity of the function f,

$$\leqq\int_0^\infty t^{-q/r-1}\sum_{n=0}^\infty f(1/\beta^n t)^q(\beta^{n+1}t)^{-2q}(\beta^n-\beta^{n+1})^q t^q$$

using a change of variables,

$$\begin{aligned}&=\sum_{n=0}^\infty\beta^{-2q}(\beta^n)^{q/r-q}(\beta^n-\beta^{n+1})^q\int_0^\infty t^{q/r+q-1}f(t)^q\,dt\\&\leqq\beta^{-2q}(1/(1-\beta^{q/r}))\int_0^\infty[t^{1/r+1}f(t)]^q\frac{dt}{t}.\end{aligned}$$

To complete the proof, choose $\beta=\frac{1}{2}$ and take the qth root of each side.

THEOREM 2.12. *Let T be a convolution operator and $h=T(f,g)$. T can be uniquely extended so that if $f\in L(p_1,q_1)$, $g\in L(p_2,q_2)$, where $1/p_1+1/p_2>1$, $1<p_i<\infty$, $0<q_i\leqq\infty$ $(i=1,2)$, then $h\in L(r,s)$, where $1/r=1/p_1+1/p_2-1$ and $s>0$ is any number such that $1/q_1+1/q_2\geqq 1/s$. Moreover*

(1) $h^{**}(t)\leqq C(tf^{**}(t)g^{**}(t)+\int_t^\infty f^*(u)g^*(u)\,du)$,

(2) $\|h\|_{(r,s)}\leqq C(r,q_1,q_2,s)\|f\|_{(p_1,q_1)}\|g\|_{(p_2,q_2)}$, *where* $C(r,q_1,q_2,s)=O(r(\alpha^{1/\alpha-1/s}))$, $1/\alpha=1/q_1+1/q_2$, *if* $s\geqq 1$, *and* $C(r,q_1,q_2,s)\leqq O(2^{s/r}-1)^{-1/s}(\alpha^{1/\alpha-1/s})$, $1/\alpha=1/q_1+1/q_2$, *if* $0<s<1$.

Proof. *Case* 1. $0<q_1<\infty$, $0<q_2<\infty$. Choose $\alpha<\infty$ such that $1/q_1+1/q_2=1/\alpha$. Then $\alpha\leqq s$. Since $\|h\|_{(r,s)}\leqq C(\alpha,s)\|h\|_{(r,\alpha)}$, where $C(\alpha,s)=O(\alpha^{1/\alpha-1/s})$ it is sufficient to assume $1/q_1+1/q_2=1/s$. The rest of the proof proceeds in the same way as in Theorem 2.4 only this time after establishing (1) we use Lemmas 2.3, 2.11 and then Hölder's inequality to establish (2).

Case 2. $q_1=\infty$, $0<q_2<\infty$ $(0<q_1<\infty, q_2=\infty)$. By the symmetry of the arguments, assume without loss of generality that $q_1=\infty$, $0<q_2<\infty$. If $1/q_2\geqq 1/s$, then $q_2\leqq s$. Since $\|h\|_{(r,s)}\leqq C(q_2,s)\|h\|_{(r,q_2)}$, where $C(q_2,s)=O(q_2^{1/q_2-1/s})$, it is sufficient to assume $s=q_2$. Note in this case $\alpha=q_2$ in the statement of the theorem. The rest of the proof proceeds as in Theorem 2.9 and Case 1 above.

The last case requires a proof differing somewhat from the preceding cases. We break the proof down into several steps.

Case 3. $q_1=q_2=\infty$ (this forces $s=\infty$).

(I) Let $f\in L(p_1,\infty)$, $g\in L(p_2,\infty)$. Put

$$\begin{aligned} f^v(x) &= f(x) \quad \text{if } |f(x)|>f^*(v),\\ &= 0 \qquad\ \text{otherwise,}\end{aligned}$$

and $f_v(x)=f(x)-f^v(x)$, where $0<v<\infty$.

It is easy to show that $f^v\in L^1$ (see Lemma 2.5). Since $g\in L(p_2,\infty)$, Theorem 2.9 implies $T(f^v,g)\in L(p_2,\infty)$.

Choose $p_{2'}$ such that $1/p_2+1/p_{2'}=1$. Thus $1/p_1=1/r+1-1/p_2=1/r+1/p_{2'}$ and $p_1<p_{2'}$. By Lemma 2.5, $f_v\in L(p_{2'},1)$. Since $g\in L(p_2,\infty)$, Theorem 2.10 implies $T(f_v,g)\in L^\infty$.

This leads us to define

$$T(f,g) = T(f_1,g)+T(f_2,g),$$

where $f=f_1+f_2$, $f_1\in L^1$, $f_2\in L(p_{2'},1)$. In the same way as before (see Lemma 2.7) it can be shown that

$$T(f^v,g)+T(f_v,g) = T(f_1,g)+T(f_2,g),$$

where $0<v<\infty$. Hence $T(f,g)$ is defined independent of any particular choices of $f_1\in L^1$, $f_2\in L(p_{2'},1)$, $f=f_1+f_2$. Moreover, the convolution operator T is bilinear on $L(p_1,\infty)\times L(p_2,\infty)$. The technique of the proof is the same as in Lemma 2.5.

(II) Using Theorems 2.9 and 2.10 we have

$$\begin{aligned} T(f, g)^{**}(t) &\leqq T(f^v, g)^{**}(t)+T(f_v, g)^{**}(t) \\ &\leqq C\left\{t(f^v)^{**}(t)g^{**}(t)+\int_t^\infty (f^v)^*(u)g^*(u)\,du \right.\\ &\qquad \left. +t(f_v)^{**}(t)g^{**}(t)+\int_t^\infty (f_v)^*(u)g^*(u)\,du\right\}. \end{aligned}$$

Since $|f^v| \leqq |f|$, $|f_v| \leqq |f|$,

$$T(f, g)^{**}(t) \leqq C\left(tf^{**}(t)g^{**}(t)+\int_t^\infty f^*(u)g^*(u)\,du\right).$$

This establishes (1).

(III) Using Lemma 2.3 we have

$$\begin{aligned} t^{1/r}h^{**}(t) &\leqq Ct^{1/r}\int_t^\infty f^{**}(u)g^{**}(u)\,du \\ &\leqq Ct^{1/r}\int_t^\infty u^{-1/p_1-1/p_2}\|f\|_{(p_1,\infty)}\|g\|_{(p_2,\infty)}\,du \\ &= rC\|f\|_{(p_1,\infty)}\|g\|_{(p_2,\infty)}. \end{aligned}$$

Therefore

$$\|h\|_{(r,\infty)} \leqq C(r, \infty, \infty, \infty)\|f\|_{(p_1,\infty)}\|g\|_{(p_2,\infty)}.$$

This establishes (2). Finally, we complete the proof with the remark that the convolution operator T is bilinear on $L(p_1, q_1)\times L(p_2, q_2)$ into $L(r, s)$ in each of the above cases.

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Department of Mathematics, Wright State University, Dayton, Ohio 45431